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EXTENSION OF AN ELASTIC SPACE WITH AN ISOLATED STIFF ROD*

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The problem of the extension of an unbounded elastic medium with an isolated rectilinear thin elastic finite rod placed rigidly therein is considered. The elastic modulus of the rod is considered to be very much larger than for the medium. An approximate asymptotic solution of the problem is proposed based on the introduction of a boundary layer in the neighbourhood of the rod; a formula is obtained for the optimal length of the bonding fiber. The boundary layer problem is then solved by the method of asymptotic Γ-integration /1/, which naturally leads to the same results. Numerical experiments on a computer, using the finite element method, are presented. One result of the numerical computation is utilized to construct an analytic solution. An estimate is given for the accuracy of the approximate analytic solution.

1. Formulation of the Problem. Let a thin rectilinear elastic rod (filament) of circular cross-section be placed in an unbounded homogeneous and isotropic elastic medium that experiences unilateral extension along the direction of the rod at infinity (Fig.1). The ideal adhesion conditions, hold at the contact boundary, i.e., all the displacement components are continuous everywhere.

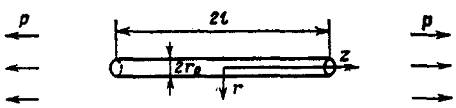

Fig.1

We introduce a cylindrical Orz coordinate system (the z axis coincides with the axis of the rod, and the point O is the midpoint of the rod). Let l be half the rod length, r_0 the radius of its cross-section, p the tensile stress σ_z at infinity, E_m, ν_m and E_f, ν_f Young's modulus and Poisson's ratio of the matrix (basic material) and rod material, respectively.

There are four positive dimensionless parameters $\nu_m, \nu_f, \lambda = r_0/l \ll 1, \varepsilon = E_m/E_f \ll 1$ in this three-dimensional problem of the theory of elasticity (by assumption the rod is thin and very much stiffer than the matrix).

The distribution along z of the mean stress $\sigma = \sigma_z$ (over the section) in the rod, and the tangential stress $\tau = \tau_{rz}$ on the lateral surfaces of the cylindrical rod, are of greater interest. They are related by the equilibrium equation

$$\tau = -{}^1/_2 r_0 d\sigma/dz$$

As $\varepsilon \to 0$ (an "inextensible" rod) the solution of this problem was first proposed in /1/ as one of the examples of applying the method of asymptotic Γ-integration. However, calculation errors were made during the solution, as Eshelby /2/ pointed out when he proposed the solution of this problem as $\varepsilon \to 0$ using the method /3/ of solving approximately the field theory problem of a conducting cylinder in the parallel electrostatic field of a dielectric.

Eshelby indicated that the method by which Van Dyke solved the electrostatic Landau and Lifshitz problem in the appendix to Taylor's paper /4/, and also the Hallen method of solving the same electrostatic problem /5/, leads to the same results.

The analytical and numerical solution of problems for small ε and λ, presented below shows, however, that for any arbitrarily small ε the quantities σ, τ and σ_{max} in a rod can be as different as desired (for instance, by one or two orders) from the values determined by the formulas in /2/, depending on the values of the other small parameter λ. Therefore, the solution of /2/ generally turns out to be unsuitable for any arbitrarily small ε.

2. Approximate analytic solution. Let us first consider the limiting case of an infinitely long rod ($\lambda = 0$). In this case, the exact solution /6/ of the Lamé equations, which satisfies the adhesion conditions for $r = r_0$ and the conditions at infinity as $r \to \infty$ by assuming, for simplicity, that $\nu_m = \nu_f$, can be written as follows (w is the displacement component along the z axis):

$$\sigma_z = \begin{cases} p/\varepsilon, & r < r_0 \\ p, & r > r_0 \end{cases}, \quad \sigma_r = \sigma_\theta = 0, \quad \frac{\partial w}{\partial z} = \frac{p}{E_m} \tag{2.1}$$

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When $\nu_m \neq \nu_f$, the errors in (2.1) for the characteristic quantities σ_z and w do not exceed several percent for all cases of practical significance. (In the compression case, when $p < 0$, the stresses σ_r and σ_θ in a fiber will be tensile for $\nu_m > \nu_f$ and, consequently, can result in splitting of the fiber and failure of the composite in certain cases, despite its small value /6/).

We will now examine the case of a rod of finite, but quite large, length when $1 \gg \lambda > 0$. The diagram of the displacement w in the sections $z = \mathrm{const}$ will recall the velocity profile in a viscous fluid boundary layer. Consequently, it is natural to make the following assumptions in a certain neighbourhood of the rod (the boundary-layer approximation):

$$\frac{\partial w}{\partial r} \gg \frac{\partial u}{\partial z}, \quad \frac{\partial^2 w}{\partial r^2} \gg \frac{\partial^2 w}{\partial z^2} \quad \sigma_r \ll \sigma_z, \quad \sigma_\theta \ll \sigma_z, \quad u \ll w$$

Omitting small terms in the equations of the theory of elasticity, in view of these assumptions, we arrive at the boundary layer equations

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} = 0, \quad \sigma_z = E_m \frac{\partial w}{\partial z} \tag{2.2}$$
$$\sigma = E_f \frac{dW}{dz}, \quad \tau = -\frac{1}{2} r_0 \frac{d\sigma}{dz}, \quad \tau_{rz} = G_m \frac{\partial w}{\partial r}$$

Here W is the rod displacement along the z axis, τ_{rz} is the tangential stress, and G_m is the shear modulus of the matrix.

The solution of (2.2) has the form

$$w = \frac{r_0 \tau}{G_m} \ln \frac{r}{r_0} + W, \quad \tau_{rz} = \tau \frac{r_0}{r} \tag{2.3}$$
$$\sigma_z = 2(1 + \nu_m) r_0 \tau'(z) \ln \frac{r}{r_0} + W'(z)$$
$$(r_0 E_f W''(z) = -2\tau)$$

For $r = r_*$ on the boundary layer boundary, the solution of (2.3) should "merge" with the unperturbed solution $\partial w / \partial z = p/E_m$ (r_* is the merger radius). We hence obtain the following equation /7/:

$$W'(z) + \frac{r_0}{G_m} \ln \frac{r_*}{r_0} \tau'(z) = \frac{p}{E_m} \tag{2.4}$$

Together with the last equation in (2.3), this serves to determine $\tau(z)$ and $W(z)$ in the boundary layer. We will write the solution of these equations that satisfies the symmetry conditions $\tau = 0, W = 0$ for $z = 0$, by selecting the "merger" radius r_* as follows:

$$\frac{r_*}{r_0} = \left(\frac{l}{r_0}\right)^\alpha = \left(\frac{1}{\lambda}\right)^\alpha \tag{2.5}$$

where α is a certain merger parameter (of the order of one). We find

$$\tau = (p - \varepsilon\sigma_{\max}) \frac{\lambda k}{2\varepsilon} \operatorname{sh} \frac{kz}{l} \tag{2.6}$$
$$\sigma = \frac{p}{\varepsilon} - \left(\frac{p}{\varepsilon} - \sigma_{\max}\right) \operatorname{ch} \frac{kz}{l}$$
$$W = \frac{p}{E_m} z - \alpha\tau(z) \frac{r_0}{G_m} \ln \frac{1}{\lambda}$$
$$\left(k^2 = \frac{\varepsilon}{\alpha\lambda^2 (1 + \nu_m) \ln (1/\lambda)}\right)$$

Here $\sigma_{\max}$ is the highest value of σ (for $z = 0$).

The stress σ at the ends of the rod is negligible; consequently, it is possible to set $\sigma = 0$ for $z = \pm l$.

Hence, by using (2.6) we find

$$\sigma_{\max} = \frac{p}{\varepsilon}\left(1 - \frac{1}{\operatorname{ch} k}\right) = \begin{cases} p/\varepsilon, & k \gg 1 \\ pk^2/(2\varepsilon), & k \ll 1 \end{cases} \tag{2.7}$$

We use the data from a numerical computation of the initial problem by the finite-element method (see Sect.4 below) to determine the parameter α. In particular, for $\varepsilon = 10^{-5}, \lambda = 10^{-2}$, and $\nu_m = 0.3$ we obtained $\sigma_{\max} = 1130p$. In this case we use (2.7) for $k \ll 1$, and we obtain $\alpha = 0.738$.

It is seen from (2.7) that as the length of the bonding rod increases the maximum length increases monotonically. The following interesting corollary results from the solution. We denote the fracture strength of the rod by σ_{st}. On the basis of (2.6) and (2.7), if

$$\lambda^2 \ln \frac{1}{\lambda} \leqslant \delta_0 \quad \left(\delta_0 = \frac{\varepsilon}{\alpha(1+\nu_m)} \operatorname{arch} \frac{p}{p-\varepsilon\sigma_{st}}\right) \tag{2.8}$$

then the bounding rod is fractured in two. The asymptotic solution of inequality (2.8) results for small λ in the explicit expression

$$l \geqslant r_0 \,[\ln(1/\delta_0)/(2\delta_0)]^{-1/2} \tag{2.9}$$

The quantity on the right side of (2.9) plays the part of an optical bonding fiber length for which its strength properties are used to the maximum extent.

3. Approximation solution by the method of asymptotic Γ-integration. Following /1/, we replace the action of the thin fiber on the matrix by the action of a lumped force of strength Z distributed along the z-axis and directed along this axis (from the equilibrium condition $Z = 2\pi\tau r_0$). We have /6, 7/

$$\int_\Sigma (Un_z - \sigma_{ij} n_j u_{i,z})\, d\Sigma = 0 \quad (i, j = 1, 2, 3) \tag{3.1}$$

Here U is the elastic potential of unit volume, σ_{ij} are stresses, u_i is the displacement, n_i is the external unit normal to the closed surface Σ comprised of two endfaces $z = \text{const}$ and two coaxial circular cylinders Σ_ε and Σ_* of identical length Δ (where $l \gg \Delta \gg r_*'$); $\Sigma_\varepsilon : r = \varepsilon_1$, where $\varepsilon_1 \leqslant r_0$; $\Sigma_* : r = r_*'$, where $l \gg r_*' \gg r_0$. The integral (3.1) is negligibly small on the endface part of the surface Σ as compared with the other components (since $\Delta \gg r_*'$) while $n_z = 0$ on the cylinders Σ_ε and Σ_*. Consequently, from (3.1) we obtain (taking account of the axial symmetry)

$$\int_{\Sigma_\varepsilon} (\tau_{rz} u_{z,z} + \sigma_r u_{r,z})\, d\Sigma = \int_{\Sigma_*} (\tau_{rz} u_{z,z} + \sigma_r u_{r,z})\, d\Sigma \tag{3.2}$$

According to the rules of Γ-integration /1, 6/ and the equilibrium equations, we have

$$\int_{\Sigma_\varepsilon} (\ldots)\, d\Sigma = \int_{\Sigma_\varepsilon} (\tau_{rz}^s u_{z,z}^0 + \sigma_r^s u_{r,z}^0)\, d\Sigma = u_{z,z}^0 \int_{\Sigma_\varepsilon} \tau_{rz}^s\, d\Sigma + u_{r,z}^0 \int_{\Sigma_\varepsilon} \sigma_r^s\, d\Sigma = Z u_{z,z}^0 |_{\varepsilon \to 0} \Delta \tag{3.3}$$

$$\int_{\Sigma_*} (\ldots)\, d\Sigma = \int_{\Sigma_*} (\tau_{rz}^0 u_{z,z}^s + \sigma_r^0 u_{r,z}^s)\, d\Sigma = u_{z,z}^s \int_{\Sigma_*} \tau_{rz}^0\, d\Sigma + u_{r,z}^s \int_{\Sigma_*} \sigma_r^0\, d\Sigma = Z u_{z,z}^s |_{r = r_*'} \Delta$$

The superscripts 0 and s refer to the regular (unperturbed) and singular (perturbed) components, respectively.

According to (3.2) and (3.3), we obtain

$$u_{z,z}^0 |_{\varepsilon \to 0} = u_{z,z}^s |_{r = r_*} \tag{3.4}$$

We obviously have

$$u_{z,z}^0 = \frac{p}{Z_m}, \quad u_{z,z}^s = \frac{Z'(z)}{2\pi G_m} \ln \frac{r}{r_0} + W'(z) \quad (E = 2\pi\tau r_0)$$

We hence obtain relationship (2.4) in conformity with (3.4). The further progress of the solution is analogous to that described in Sect.2.

The selfinduced component in the expression for u_z^0, equal to

$$\frac{1+\nu_m}{2\pi E_m} \int_{l_*}^{l} \frac{Z(t) - Z(z)}{|t-z|}\, dt$$

is omitted as being negligibly small compared with pz/E_m.

4. Computation by the finite-element method. To confirm the relationships obtained, computations were performed by using a perfected version of the NEPTUN program /8/.

A cylinder of radius R_0 and altitude $2L$, in which a cylindrical rod of radius r_0 and length $2l$ is palced, was used as the computational scheme to solve the problem. The cylinder (matrix) and rod materails have different elastic characteristics. The following dimensions were taken for the computation: $r_0 = 0.5$, $l = 50$, $R_0 = 100$, $L = 150$ ($\lambda = 10^{-2}$).

By virtue of symmetry, the upper half of the cylinder with the displacement and stress boundary conditions $w = 0$ for $z = 0$ and $\sigma_z = p$ for $z = L$ was considered in the solution.

Isoparametric quadratic elements with eight nodes were used to construct the discrete model in the rz plane. The finite-element mesh is generated automatically by using the parametric assignment of coordinates. It consists of 176 elements and has 1166 degees of freedom. The mesh is made denser in the domain adjoining the rod.

The results of computing the normal stress σ in the rod and the tangential stress τ on the rod-matrix boundary are presented in Fig.2a and b for different $\varepsilon = E_m/E_f$ and are shown by the dashed lines $1-5$ corresponding to the values $\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$. It was assumed here that Poisson's ratios are $\nu_m = \nu_f = 0.3$. The same quantities computed by means of (2.6) are shown by the solid lines.

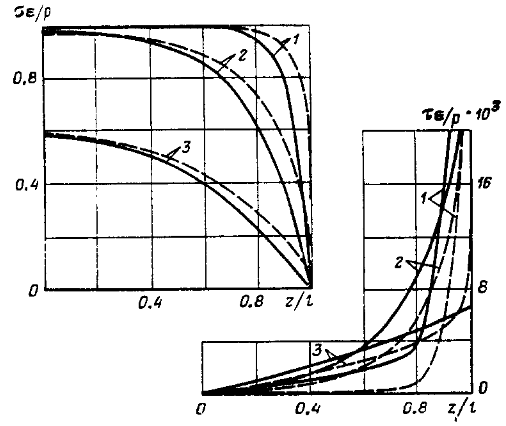

Fig.2a

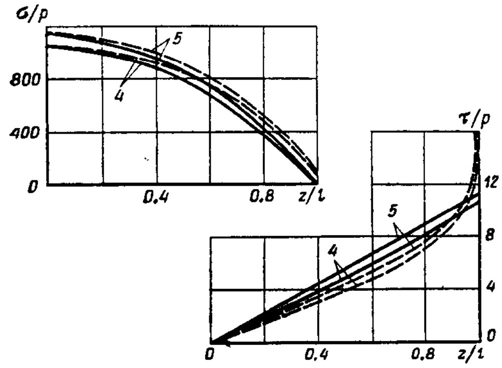

Fig.2b

Good agreement is observed between the numerical and analytical relations for the normal stresses in the rod. The maximum value σ_{max} is predicted almost exactly by (2.6). The greatest differences in σ are observed on approaching the end of the rod, which is due to neglecting the normal stress at the ends of the rod in deriving (2.6). The differences between the computed and analytical values of τ are greater, as is natural, since the tangential stress is a derivative of σ. Nevertheless, these differences are not large for small ε.

To verify the influence of the matrix size (the rod dimensions are unchanged) on the solution obtained by the finite-element method, repeated computations were made with $R_0 = 200$, $L = 250$. For $\varepsilon = 10^{-5}$ an increase of less than 2% was obtained in the stress.

A complete numerical investigation of the problem as a whole is difficult because there are four dimensionless free parameters $\varepsilon, \lambda, \nu_m$ and ν_f in the problem. Numerical computations were performed for $\varepsilon = 0$ (absolutely rigid rod) with $\nu_{m1} = 0.3$ and $\nu_{m2} = 0.49$. From the analytic solution

$$\frac{\sigma_{\max 1}}{\sigma_{\max 2}} = \frac{1+\nu_{m2}}{1+\nu_{m1}} = 1.146.$$

This quantity equals 1.110 according to the results of the numerical computations.

Sects.1—3 were written by G. P. Cherepanov, and Sect.4 by G. P. Nikishkov.

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